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COMMENT

Quasilinearisation and three-dimensional oscillators

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Abstract. The anharmonic symmetric oscillator is a frequently used model for physical systems. The recently developed quasilinearisation technique is applied to this model as well as to the simpler one without the quartic term in the potential.

1. Motivation

The quasilinearisation technique of Bellman and Kalaba [1] has recently been applied [2] to the WKB solution of the Schrödinger equation with the Coulomb potential. Since this method is rather transparent as well as quite powerful, it is of interest to extend its study to other physical systems. In this comment, we shall first test the procedure on the exactly solvable isotropic harmonic oscillator, and then apply it to the anharmonic symmetric oscillator which has recently received considerable attention in the literature.

2. The method

The radial Schrödinger equation for a spherically symmetric potential $V(r)$ is customarily written as

$$u''(r)+\lambda^2k^2u(r)=0 \tag{1}$$

where

$$\lambda^2=2m/\hbar^2 \qquad k^2=E-V(r)-L^2/\lambda^2r^2. \tag{2}$$

The physical value of L^2 is $l(l+1)$, but it is well known that somewhat different values need to be taken in various orders of the WKB approximation in order to ensure the correct behaviour of the wavefunction, namely $u\sim r^{l+1}$, near the origin. When all orders are summed up, the actual value $l(l+1)$ is reproduced [3]. Keeping this in mind, it is convenient to treat L^2 as an adjustable parameter in the approximation scheme.

The usual WKB substitution is now made:

$$u(r)=\text{constant}\times\exp\left(\lambda\int y(r)\,\mathrm{d}r\right). \tag{3}$$

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The function $y(r)$ is seen to obey the equation

$$y' = -\lambda (y^2 + k^2). \tag{4}$$

Introducing a new variable

$$x = \lambda r \tag{5}$$

equation (4) simplifies to

$$y'(x) = -y^2(x) - k^2(x). \tag{6}$$

Applying the quasilinearisation technique to this non-linear equation, the following recursion differential equations are obtained [2]:

$$y'_{p+1} = y_p^2 - 2y_p y_{p+1} - k^2. \tag{7}$$

These equations may be successively solved for higher values of p so as to obtain the desired order of approximation to the true solution of (6). The lowest- (zeroth-) order solution is given by

$$y_0(x) = \mathrm{i}k(x). \tag{8}$$

The next approximation $y_1(x)$ satisfies the equation

$$y_1' = -2\mathrm{i}ky_1 - 2k^2. \tag{9}$$

This can be integrated to obtain

$$y_1(x) = \sum_{n=0}^{\infty} y_1^{[n]}(x) \tag{10}$$

where

$$y_1^{[n+1]} = \frac{1}{2\mathrm{i}k}\frac{\mathrm{d}}{\mathrm{d}x}(-y_1^{[n]}) \qquad y_1^{[0]} = \mathrm{i}k. \tag{11}$$

The functions $y_1^{[n]}$ can be explicitly calculated to yield the series

$$y_1(x) = \mathrm{i}k - \frac{k'}{2k} + \frac{1}{4\mathrm{i}k}\left(\frac{k''}{k} - \frac{k'^2}{k^2}\right) + \frac{1}{8k}\left(\frac{k'''}{k^2} - \frac{4k'k''}{k^3} + \frac{3k'^3}{k^4}\right) + \dots. \tag{12}$$

We note in passing that the corresponding equation (3.12) of [2] contains several errors, presumably because of misprinting.

Using $y_1(x)$ from (12) as input, higher-order approximations can be obtained from (7), but detailed analysis is restricted to this order in the present study.

3. Harmonic oscillator

The potential energy function for an isotropic harmonic oscillator may be written as

$$V(r) = \tfrac{1}{2}m\omega^2 r^2. \tag{13}$$

Using the quantities λ and x defined previously, and introducing another parameter

$$\varepsilon = \tfrac{1}{2}\hbar\omega \tag{14}$$

the potential energy can be expressed as

$$V(x) = \varepsilon^2 x^2. \tag{15}$$

The function k^2 of (2) then becomes

$$k^2(x) = E - \varepsilon^2 x^2 - L^2/x^2 \tag{16}$$

or

$$k = P(x)/x \tag{17a}$$

with

$$P(x) = (-\varepsilon^2 x^4 + Ex^2 - L^2)^{1/2}. \tag{17b}$$

The classical turning points, obtained by equating k^2 to zero, are

$$x = \frac{1}{\varepsilon}\left(\frac{E \pm (E^2 - 4\varepsilon^2 L^2)^{1/2}}{2}\right)^{1/2}. \tag{18}$$

From the reality condition on x, we get the following lower bound on energy:

$$E \geqslant \hbar\omega L. \tag{19}$$

The 'quasilinearised' (7) can now be solved explicitly, to obtain for the harmonic oscillator:

$$y_1(x) = \frac{\mathrm{i}P}{x} + \frac{\varepsilon^2 x^4 - L^2}{2xP^2} + \frac{\mathrm{i}}{4xP} + \frac{\mathrm{i}(8\varepsilon^2 x^4 - 3Ex^2 + 2L^2)}{4xP^3} + \frac{\mathrm{i}x^3(2\varepsilon^2 x^2 - E)^2}{2P^5} + \ldots. \tag{20}$$

Substituting y_1 in (3), we obtain the radial wavefunction for the harmonic oscillator, correct to the first two orders. It can easily be checked that the proper behaviour of $u(r)$ for large distances is reproduced:

$$u(r) \underset{r\to\infty}{\sim} \exp\left(-\frac{m\omega}{2\hbar} r^2\right). \tag{21}$$

Energy eigenvalues are obtained by using the quantisation condition

$$2\pi \mathrm{i} n = \oint y \,\mathrm{d}x \tag{22}$$

where the integral is taken around the branch cut joining the two classical turning points given by (18). Using residues, the integral can easily be evaluated to yield

$$2\pi \mathrm{i} n = -2\pi \mathrm{i}(L+1) + \mathrm{i}\pi E/\varepsilon$$

or

$$E = 2(n + L + 1)\varepsilon.$$

It is shown in [3] that, in the order considered, the parameter L^2 has the value $(l + \frac{1}{2})^2$. Hence

$$E = (n + l + \tfrac{3}{2})\hbar\omega \tag{23}$$

which is the right formula for the energy spectrum of a three-dimensional oscillator, remembering that n should be an even integer from parity considerations.

4. Anharmonic oscillator

We now consider the anharmonic symmetric oscillator which has recently been studied [4, 5]. The potential for the system may be taken as:

$$V(r)=\tfrac{1}{2}m\omega^2r^2+\alpha\lambda^4r^4. \tag{24}$$

In terms of the variable x introduced previously, the potential can be written as

$$V(x)=\varepsilon^2x^2+\alpha x^4. \tag{25}$$

The corresponding function k^2 is then

$$k^2(x)=E-\varepsilon^2x^2-\alpha x^4-L^2/x^2. \tag{26}$$

Hence

$$k(x)=Q(x)/x \tag{27a}$$

with

$$Q(x)=(-\alpha x^6-\varepsilon^2x^4+Ex^2-L^2)^{1/2}. \tag{27b}$$

The equation $k^2=0$ can be solved as a cubic algebraic equation in x^2. It turns out that, in general, there are two positive, two negative and two complex roots. The classical turning points correspond to the positive square roots of the expressions

$$\frac{1}{\alpha}\left[2H\cos\left(\frac{2n\pi+\theta}{3}\right)-\frac{\varepsilon^2}{3}\right] \qquad \text{where} \qquad \cos\theta=-G/2H^3$$

and

$$G=\tfrac{1}{27}(2\varepsilon^6+9\alpha E\varepsilon^2+27\alpha^2L^2)$$

$$H=\tfrac{1}{3}(\varepsilon^4+3\alpha E)^{1/2}.$$

As in the case of the harmonic oscillator, the function $y_1(x)$ is obtained by calculating the derivatives of $k(x)$ and using (12). Explicitly, we obtain

$$y_1(x)=\frac{\mathrm{i}Q}{x}+\frac{2\alpha x^6+\varepsilon^2x^4-L^2}{2xQ^2}+\frac{\mathrm{i}}{4xQ}+\frac{\mathrm{i}(17\alpha x^6+8\varepsilon^2x^4-3Ex^2+2L^2)}{4xQ^3}$$
$$+\frac{\mathrm{i}x^3(3\alpha x^4+2\varepsilon^2x^2-E)^2}{2Q^5}+\ldots \tag{28}$$

where Q is given by (27b).

When $\alpha=0$, the solution for the harmonic oscillator is, of course, recovered.

5. Conclusion

It has been shown that the quasilinearisation technique works very well for the spherically symmetric three-dimensional oscillator. This enhances confidence in the procedure which is then applied to the harmonic oscillator potential perturbed by a quartic term.

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References

[1] Bellman R and Kalaba R 1975 *Quasi-linearisation and Non-linear Problems* (Amsterdam: Elsevier)
[2] Ragunathan K and Vasudevan R 1987 *J. Phys. A: Math. Gen.* **20** 839
[3] Seetharaman M and Vasan S S 1984 *J. Phys. A: Math. Gen.* **17** 2485
[4] Sanches A M and Bejarano J D 1986 *J. Phys. A: Math. Gen.* **19** 887
[5] Killingbeck J 1987 *J. Phys. A: Math. Gen.* **20** 601